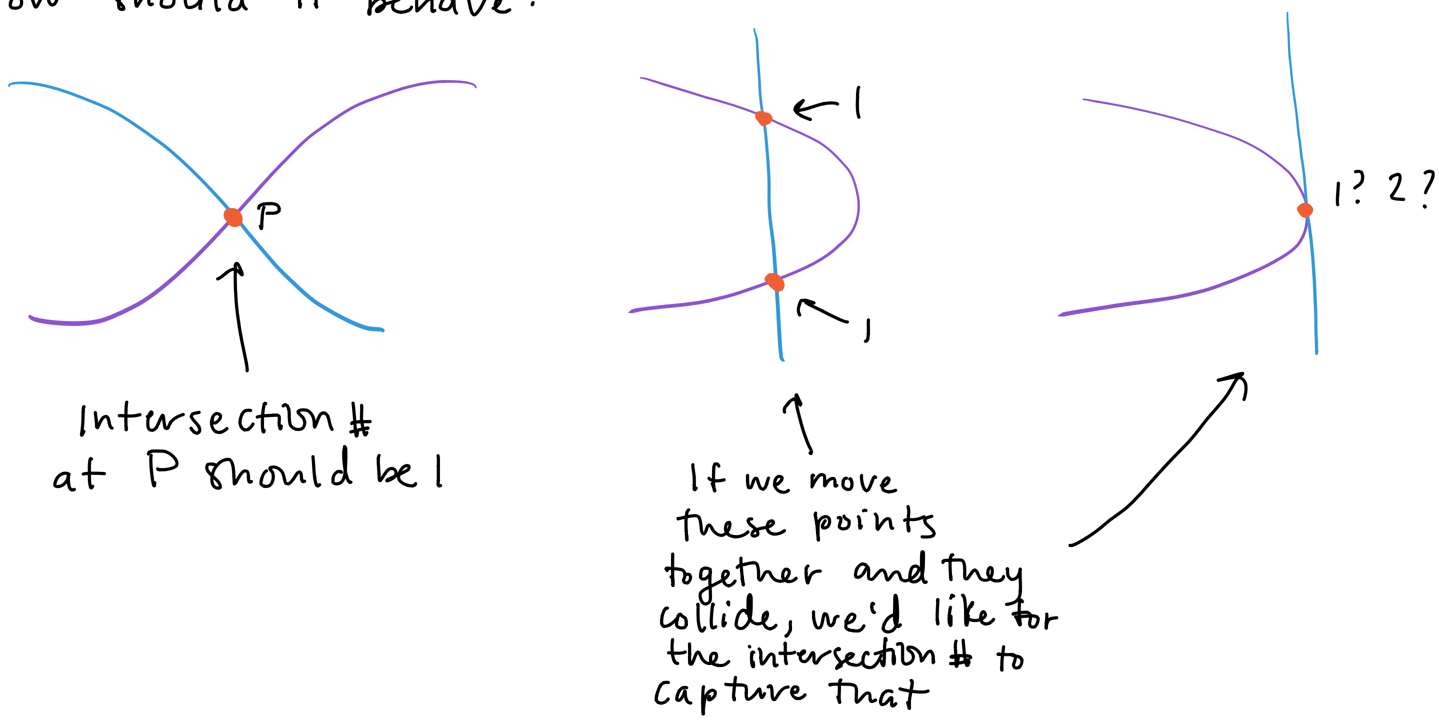
Intersection numbers

Let f, g be plane curves and $P \in \mathbb{A}^2$. We want to define an intersection # of f and g at P, $I_P(f,g)$.

How should it behave?



First we'll list 7 properties we want $I_P(f,g)$ to have. Then, we'll show the properties define a unique such #.

Properties of $I_P(f,g)$

1.) If f and g have a common component that passes through P, then $I_P(f,g) = \infty$. Otherwise $I_P(f,g) \in \mathbb{Z}_{\geq 0}$.

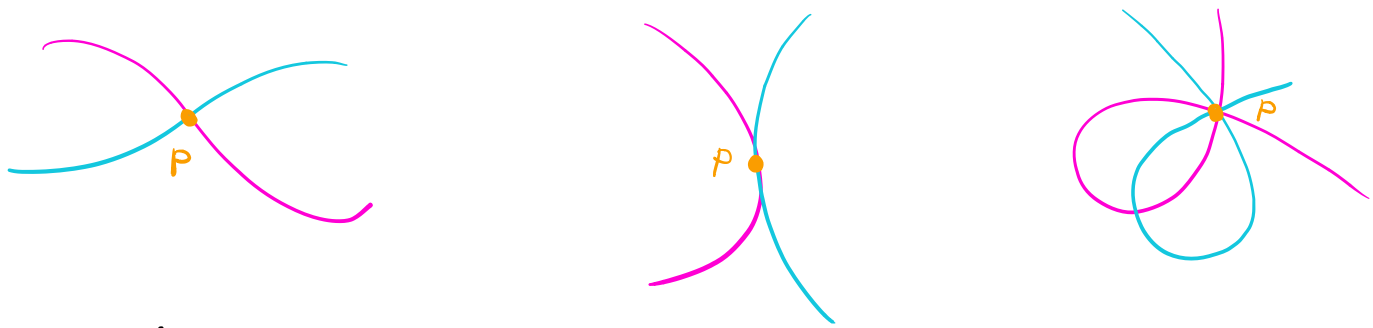
2.) a.) $I_P(f,g) = 0 \iff P \notin V(f) \cap V(g)$.

b.) $I_P(f,g)$ depends only on components of f and g that pass through P.

3.) If T is an affine change of coordinates and $T(Q) = P$, then $I_P(f,g) = I_Q(T^*f, T^*g)$

4.) $I_P(f,g) = I_P(g,f)$

5.) $I_P(f,g) \geq m_P(f)\, m_P(g)$, with equality if and only if f and g have no tangent lines in common at P.



$I_P(f,g) = 1$ $\qquad$ $I_P(f,g) > 1$ $\qquad$ $I_P(f,g) = 2\cdot 2 = 4$

Def: f and g <u>intersect transversally</u> at P if P is a simple point on both f and g and the tangent lines are distinct. (i.e. iff $I_P(f,g) = 1$.)

6.) If $f = \prod f_i^{r_i}$, $g = \prod g_j^{s_j}$, then $I_P(f,g) = \sum_{i,j} r_i s_j I_P(f_i, g_j)$.

7.) $I_P(f,g) = I_P(f, g + af)$ for $a \in k[x,y]$, i.e. the intersection # depends only on the image of g in $\Gamma(f)$.

Lemma: Any intersection # satisfying these properties is unique.

Pf: Assume $I_p(f,g)$ is a # satisfying 1.) - 7.). We give a procedure for calculating it:

By 3.), we can assume $P=(0,0)$, and by 1.), we can assume it's finite. By 2.) we know when $I_p(f,g)=0$.

Assume $I_p(f,g)=n>0$, and $I_p(a,b)$ can be calculated using these properties for $I_p(a,b)<n$.

Consider $f(x,0), g(x,0) \in k[x]$ of degrees r, s, resp. (set r or $s=0$ if the corr. polynomial is zero)

By 4.), we can assume $r \leq s$.

Case 1: $r=0$. Then $y \mid f$, so $f=yh$.

$$\Rightarrow I_p(f,g) = \underset{\substack{\vee \\ 0}}{I_p(y,g)} + \underset{\substack{\wedge \\ n}}{I_p(h,g)}$$

since $P \in V(g)$ $\Rightarrow$ which can be calculated by induction

What is the first summand?

Write $g(x,0) = x^m(a_0 + a_1 x + \cdots)$, $a_0 \neq 0$.

$$\Rightarrow g = Ax^m + By, \quad A(P) \neq 0.$$

$$\Rightarrow I_p(g,y) \underset{7.)}{=} I_p(Ax^m, y) \underset{6.)}{=} \underbrace{I_p(A,y)}_{0 \leftarrow 2.)} + m\underbrace{I_p(x,y)}_{1 \leftarrow 5.)} = m.$$

Case 2: $r>0$. WLOG $f(x,0)$ and $g(x,0)$ are monic.

Set $h = g - x^{s-r} f$. Then $h(x,0) = \underset{\substack{\uparrow \\ \text{leading term} \\ x^s}}{g(x,0)} - x^{s-r} \underbrace{f(x,0)}_{\substack{\uparrow \\ \text{leading term} \\ x^r}}$

so $\deg(h(x,0)) < s$ and $I_p(f,g) = I_p(f,h)$.

Repeating this process finitely many times (switching the order of the curves when necessary), we end up in case 1. $\square$

Thm: There is a unique intersection # satisfying 1.) - 7.), given by

$$I_p(f,g) = \dim_k\left(\mathcal{O}_P(\mathbb{A}^2)/(f,g)\right)$$

Pf: We already showed uniqueness, so we just need to show it satisfies the properties.

2.) a.) $P \notin V(f) \cap V(g) \iff \exists\ h \in (f,g) \subseteq k[x,y]$ w/ $h(P) \neq 0$.

$\iff \exists$ a unit in $\left(\frac{f}{1}, \frac{g}{1}\right) \subseteq \mathcal{O}_p(\mathbb{A}^2)$

$\iff \dim\left(\mathcal{O}_P(\mathbb{A}^2)/(f,g)\right) = 0$.

b.) If $f = f_1 f_2$ and f_2 doesn't pass through P, then f_2 is a unit in $\mathcal{O}_P(\mathbb{A}^2)$

so $(f,g) = (f_1, g) \subseteq \mathcal{O}_p(\mathbb{A}^2)$

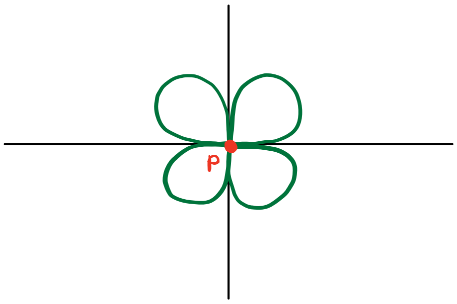
4.) Obvious.

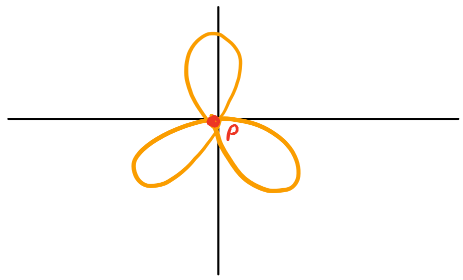
7.) $(f,g) = (f, g+af)$.

3.) An affine change of coordinates induces an isomorphism of local rings, so this holds.

For 1.), 5.), and 6.), see Fulton. $\square$

Ex: $f = (x^2+y^2)^3 - 4x^2y^2$ and $g = (x^2+y^2)^2 + 3x^2y - y^3$, $P = (0,0)$.





What is $I_P(f,g)$?

Notice: $$f - (x^2+y^2)g = -4x^2y^2 - (3x^2y - y^3)(x^2+y^2)$$
$$= y\underbrace{\left(-4x^2y - (3x^2-y^2)(x^2+y^2)\right)}_{h}$$

So $I_P(f,g) = I_P(g,h) + I_P(g,y)$

Now we continue by using the method from the proof:

h has highest deg x-term $-3x^4$, and g has x^4.

So we replace h by $h + 3g$
$$= -4x^2y - 3x^4 - 2x^2y^2 + y^4 + 3(x^4 + 2x^2y^2 + y^4 + 3x^2y - y^3)$$
$$= 5x^2y + 4x^2y^2 + 4y^4 - 3y^3$$
$$= y\underbrace{(5x^2 + 4x^2y + 4y^3 - 3y^2)}_{q}$$

$\Rightarrow I_p(f,g) = I_p(g,q) + 2I_p(g,y)$

tangent lines of q are $\sqrt{5}x \pm \sqrt{3}y$, so q and g have no tangent lines in common, so

$$I_p(g,q) = m_p(g)\cdot m_p(q) = 3\cdot 2 = 6.$$

$I_p(g,y) = I_p(x^4, y) = 4$, so $I_p(f,g) = 6 + 2\cdot 4 = 14.$

One more property of the intersection number:

Prop: If P is a simple point of f, then $I_p(f,g) = \mathrm{ord}_p^f(g)$

Pf: We can assume f is irreducible (by forgetting other components).

Since $\mathcal{O}_p(f)$ is a DVR, $\mathrm{ord}_p^f(g) = \dim_k\left(\mathcal{O}_p(f)/(g)\right)$.

$$\mathcal{O}_p(\mathbb{A}^2)/(f) \cong \mathcal{O}_p(f), \text{ so } \mathcal{O}_p(\mathbb{A}^2)/(f,g) \cong \mathcal{O}_p(f)/(g). \quad \square$$